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The formulation of gauge-Higgs unification with dynamical boundary conditions

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Abstract

The boundary conditions on multiply connected extra dimensions play major roles in gauge-Higgs unification theory. Different boundary conditions, having been given in ad hoc manner so far, lead to different theories. To solve this arbitrariness problem of boundary conditions, we construct a formulation of gauge-Higgs unification with dynamics of boundary conditions on $M^4 \times S^1/Z_2$. As a result, it is found that only highly restricted sets of boundary conditions, which lead to nontrivial symmetry breaking, practically contribute to the partition function. In particular, we show that for $SU(5)$ gauge group, sets of boundary conditions which lead to $SU(5) \to SU(3) \times SU(2) \times U(1)$ symmetry breaking are naturally included in the restricted sets.

1. Introduction

Gauge-Higgs unification (GHU) unifies gauge fields and Higgs scalar fields by considering gauge theory on higher dimensions [1,2]. When multiply connected manifolds are introduced, dynamics of Wilson line phases lead to breakdown of gauge symmetry imposed on Lagrangian density. By using this Hosotani mechanism, GHU has been extensively investigated. There arise some difficulties for GHU due to introducing higher dimensions. One of them is the chiral fermion problem. One way to solve this problem is provided by considering GHU on orbifold. Furthermore, one can get natural solution for Higgs doublet-triplet mass splitting problem in $SU(5)$ grand unified theory (GUT) [3,4]. Also, the possibilities that one might achieve the unification of three families of quarks and leptons in higher-dimensional GUT on an orbifold have

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been proposed [5,6]. But, in formulating GHU on orbifold, there remains subtlety that should be solved. At the moment, one imposes boundary conditions on multiply connected manifolds by hand, although there are a lot of possible boundary conditions imposed manifolds. We refer to this subtlety as arbitrariness problem of boundary conditions [7]. This arbitrariness problem for GHU on orbifold was investigated by N. Haba, M. Harada, Y. Hosotani, Y. Kawamura in detail [8,9]. They classified equivalence classes for boundary conditions with using Hosotani mechanism, and analyzed their physics for each equivalence class. But, to solve this problem completely, we need dynamics of boundary conditions. Then, we must understand more fundamental theory to give this dynamics.

In this paper, we treat the boundary conditions as dynamical values, not those given by hand. For this goal, we have to generalize the present GHU formulation. We need the methods by which we can analyze systematically all possible configurations for the boundary conditions in one framework. By using the matrix model analysis, we construct this framework. In this framework, we mainly focus on the natures of measures on integrations over the boundary conditions, to prove that only restricted sets of boundary conditions can contribute to the partition function, although we sum over all possible configurations for the boundary conditions. This restriction is common property in our formulation, irrespective of a detail of the action, and leads to the nontrivial gauge symmetry breaking. In particular, in the case of $SU(5)$ gauge group, the gauge-Higgs unification scenario is naturally restricted to only a few equivalence classes by the boundary conditions dynamics. Then, the equivalence class with standard model symmetry $SU(3) \times SU(2) \times U(1)$ as symmetry of boundary conditions is naturally included.

In Section 2, we give basic knowledge for GHU on orbifold, and classify each set of boundary conditions to equivalence classes. In Section 3, we give the formulation of GHU with dynamics of boundary conditions. In Section 4, our formulation is applied to several examples. Section 5 is devoted to conclusions.

2. Basic knowledges of GHU on S^1/Z_2

2.1. Boundary conditions on S^1/Z_2

In this paper, we restrict our attention to GHU on $M^4 \times S^1/Z_2$. The physics for this model was analyzed in Refs. [8,9]. M^4 is four-dimensional Minkowski spacetime. The fifth dimension S^1/Z_2 is obtained by identifying two points on S^1 by parity. Let x and y be coordinates of M^4 and S^1, respectively. S^1 has a radius R. In other words, a point $(x, y + 2\pi R)$ is identified with a point (x, y). The orbifold $M^4 \times S^1/Z_2$ is obtained by identifying $(x, y) \sim (x, y + 2\pi R) \sim (x, -y)$.

As a general principle the Lagrangian density has to be single-valued and gauge invariant on $M^4 \times S^1/Z_2$. After a loop translation along S^1, each field needs to return to its original value only up to a global transformation of $U \in G$, where G is unitary gauge group imposed on Lagrangian density. It is called S^1 boundary condition. For gauge field A_M $(M = 0 \sim 3, 5)$

$$A_M(x, y + 2\pi R) = U A_M(x, y) U^\dagger. \tag{2.1}$$

The Z_2-parity is specified by parity matrices. Around $y = 0$

$$\begin{pmatrix} A_\mu(x, -y) \\ A_y(x, -y) \end{pmatrix} = P_0 \begin{pmatrix} A_\mu(x, y) \\ -A_y(x, y) \end{pmatrix} P_0^\dagger \tag{2.2}$$

and around $y = \pi R$

$$\begin{pmatrix} A_\mu(x,\pi R-y) \\ A_y(x,\pi R-y) \end{pmatrix} = P_1 \begin{pmatrix} A_\mu(x,\pi R+y) \\ -A_y(x,\pi R+y) \end{pmatrix} P_1^\dagger. \tag{2.3}$$

To preserve the gauge invariance, A_y must have an opposite sign relative to A_μ under these transformations. As the repeated Z_2-parity operation brings a field configuration back to the original, P_0 must satisfy $P_0^2 = 1$. This means $P_0^\dagger = P_0$. P_0 must be an element of G up to an overall sign. This sign does not affect the result below so that we drop it in the following discussions. The same conditions apply to P_1, that is,

$$P_0^2 = P_1^2 = 1 \tag{2.4}$$

Among U, P_0 and P_1, the relation

$$U = P_1 P_0 \tag{2.5}$$

is satisfied.

For scalar fields, the boundary conditions are specified by

$$\begin{aligned} &\phi(x,-y) = \pm T_\phi[P_0]\phi(x,y) \\ &\phi(x,\pi R-y) = \pm e^{i\pi\beta_\phi} T_\phi[P_1]\phi(x,\pi R+y) \\ &\phi(x,y+2\pi R) = e^{i\pi\beta_\phi} T_\phi[U]\phi(x,y). \end{aligned} \tag{2.6}$$

$T_\phi[U]$ represents an appropriate representation matrix. The relation $T_\phi[U] = T_\phi[P_1]T_\phi[P_0]$ is also satisfied just as in (2.5). There are arbitrariness in the sign if the whole interaction terms in the Lagrangian remain invariant. $e^{i\pi\beta_\phi}$ must be either $+1$ or -1 due to Z_2-parity.

For Dirac fields, the boundary conditions are represented by

$$\begin{aligned} &\psi(x,-y) = \pm T_\psi[P_0]\gamma^5\psi(x,y) \\ &\psi(x,\pi R-y) = \pm e^{i\pi\beta_\psi} T_\psi[P_1]\gamma^5\psi(x,\pi R+y) \\ &\psi(x,y+2\pi R) = e^{i\pi\beta_\psi} T_\psi[U]\psi(x,y). \end{aligned} \tag{2.7}$$

The phase factor $e^{i\pi\beta_\psi}$ must be either $+1$ or -1 just as for scalar fields. $(\gamma^5)^2 = 1$ in our convention.

Therefore, the boundary conditions on $M^4 \times S^1/Z_2$ are specified with (P_0, P_1, U, β) and additional signs in (2.6) and (2.7). It is worthwhile to stress that the eigenvalues of P_0, P_1 must be either $+1$ or -1 due to the condition $P_0^2 = P_1^2 = 1$.

Next, we consider a gauge transformation on our system. Under a gauge transformation $\Omega(x,y)$, the fields change to

$$\begin{aligned} &A_M(x,y) \to A'_M(x,y) = \Omega(x,y)A_M(x,y)\Omega^\dagger(x,y) - \frac{i}{g}\Omega(x,y)\partial_M\Omega^\dagger(x,y), \\ &\phi(x,y) \to \phi'(x,y) = T_\phi\big[\Omega(x,y)\big]\phi, \qquad \psi(x,y) \to \psi'(x,y) = T_\psi\big[\Omega(x,y)\big]\psi. \end{aligned} \tag{2.8}$$

Generally, gauge transformations also change the given boundary conditions. After gauge transformation, the new fields A'_M satisfy, instead of (2.1), (2.2) and (2.3),

$$\begin{aligned} &A'_M(x,y+2\pi R) = U'A'_M(x,y)U'^\dagger - \frac{i}{g}U'\partial_M U'^\dagger \\ &\begin{pmatrix} A'_\mu(x,-y) \\ A'_y(x,-y) \end{pmatrix} = P'_0 \begin{pmatrix} A'_\mu(x,y) \\ -A'_y(x,y) \end{pmatrix} P'^\dagger_0 - \frac{i}{g}P'_0 \begin{pmatrix} \partial_\mu \\ -\partial_y \end{pmatrix} P'^\dagger_0 \\ &\begin{pmatrix} A'_\mu(x,\pi R-y) \\ A'_y(x,\pi R-y) \end{pmatrix} = P'_1 \begin{pmatrix} A'_\mu(x,\pi R+y) \\ -A'_y(x,\pi R+y) \end{pmatrix} P'^\dagger_1 - \frac{i}{g}P'_1 \begin{pmatrix} \partial_\mu \\ -\partial_y \end{pmatrix} P'^\dagger_1 \end{aligned} \tag{2.9}$$

where,

$$\begin{aligned} U' &= \Omega(x, y + 2\pi R) U \Omega^\dagger(x, y) \\ P'_0 &= \Omega(x, -y) P_0 \Omega^\dagger(x, y) \\ P'_1 &= \Omega(x, \pi R - y) P_1 \Omega^\dagger(x, \pi R + y). \end{aligned} \tag{2.10}$$

Scalar and fermion fields ϕ' and ψ' satisfy relations similar to (2.6), (2.7), where (P_0, P_1, U) is replaced by (P'_0, P'_1, U').

The gauge transformations which preserve the given boundary conditions are regard as the residual gauge invariance on the system. These transformations which satisfy $U' = U$, $P'_0 = P_0$ and $P'_1 = P_1$ are defined by

$$\begin{aligned} &\Omega(x, y + 2\pi R) U = U \Omega(x, y) \\ &\Omega(x, -y) P_0 = P_0 \Omega(x, y) \\ &\Omega(x, \pi R - y) P_1 = P_1 \Omega(x, \pi R + y). \end{aligned} \tag{2.11}$$

Eq. (2.11) is called the symmetry of boundary conditions. Note that the physical symmetry can differ from the symmetry of boundary conditions. When we consider the symmetry at low energies, namely gauge potential is independent on y: $\Omega = \Omega(x)$, the symmetry of boundary conditions is reduced to

$$\Omega(x) U = U \Omega(x), \qquad \Omega(x) P_0 = P_0 \Omega(x), \qquad \Omega(x) P_1 = P_1 \Omega(x). \tag{2.12}$$

That is, the symmetry is generated by generators which commute with U, P_0 and P_1.

We must regard theories with different boundary conditions as theories with different physical content. But theories with different boundary conditions can be equivalent in physical content. If gauge transformation defined by (2.8) satisfies the conditions

$$\partial_M P'_0 = 0, \qquad \partial_M P'_1 = 0, \qquad \partial_M U' = 0, \tag{2.13}$$

the two sets of boundary conditions are equivalent. We represent it as

$$\left(P'_0, P'_1, U'\right) \sim (P_0, P_1, U). \tag{2.14}$$

The conditions (2.13) lead to $P'^\dagger_0 = P'_0$, $P'^\dagger_1 = P'_1$. This (P'_0, P'_1, U') also satisfy (2.4), (2.5), where (P_0, P_1, U) is replaced by (P'_0, P'_1, U'). The relation (2.14) defines equivalence classes, and the two theories in the same equivalence class lead to the same physical content although these theories may have different symmetries of boundary conditions. This equivalence of physical content is ensured by the Hosotani mechanism. This mechanism plays a major role in analyzing GHU.

2.2. *Hosotani mechanism*

The Hosotani mechanism that states theories in the same equivalence class lead to the same physical content takes place by the dynamics of Wilson line phases. The Hosotani mechanism in gauge theory defined on multiply connected manifolds is described by following statement [2].

We give WU defined by

$$WU = \mathcal{P}\exp\left\{ig\int_C dy\, A_y\right\}U. \tag{2.15}$$

The phases of WU are called Wilson line phases. U is the boundary condition of loop translation along non-contractible loop, C is non-contractible loop, $\mathcal{P}$ denotes path ordered product. The eigenvalues of WU are gauge invariant, so that these phases cannot be gauged away. Therefore, we should regard WU as physical degrees of freedom. Wilson line phases are determined by dynamics of $(A_y^a, \frac{1}{2}\lambda^a \in \mathcal{H}_W)$, where

$$\mathcal{H}_W = \left\{\frac{\lambda^a}{2};\ \{\lambda^a, P_0\} = \{\lambda^a, P_1\} = 0\right\}. \tag{2.16}$$

That is, $\mathcal{H}_W$ is a set of generators which anti-commute with P_0, P_1.
Vacua of the system can degenerate at the classical level, but in general, the degeneracy of vacua is lifted by quantum effects. The vacuum given by the configuration of Wilson line phases, which minimizes the effective potential V_{eff}, becomes the physical vacuum of the system. If Wilson line phases have non-trivial configuration, the gauge symmetry imposed on system is spontaneously broken or restored by radiative corrections. As a result, gauge fields in lower dimension whose gauge symmetry is broken acquire masses from non-vanishing expectation values of the Wilson line phases. Some of matter fields also acquire masses.
Two sets of boundary conditions which can be related by a boundary-condition-changing gauge transformation are physically equivalent, even if the symmetries of boundary conditions are different. This defines equivalence classes for boundary conditions. The physical symmetry of theory depends on the matter content of the theory through the expectation values of the Wilson line phases. One can determine the physical symmetry of theory by the combination of boundary conditions and the expectation values of the Wilson line phases.

We can determine physical symmetry of the theory under given boundary conditions (P_0, P_1, U), by using this Hosotani mechanism. We suppose V_{eff} is minimized by constant $\langle A_y\rangle$, and $\exp(i2\pi gR\langle A_y\rangle) \neq I$. Then, $\langle A_y\rangle$ is transformed to $\langle A'_y\rangle = 0$ by gauge potential $\Omega(x, y) = \exp\{ig(y+\alpha)\langle A_y\rangle\}$. After this transformation, boundary conditions change to

$$\left(P_0^{sym}, P_1^{sym}, U^{sym}, \beta\right) \equiv \left(e^{2ig\alpha\langle A_y\rangle}P_0, e^{2ig(\alpha+\pi R)\langle A_y\rangle}P_1, WU, \beta\right). \tag{2.17}$$

As only extra dimensional components A_y whose generators anti-commutate with P_0, P_1 can have non-vanishing expectation values, the boundary conditions (2.17) indeed satisfy (2.4), (2.5). As $\langle A'_y\rangle = 0$ in this gauge, physical symmetry of the theory agrees with the symmetry of boundary conditions. Then, physical symmetry of theory is determined by

$$H^{sym} = \left\{\frac{\lambda^a}{2};\ \left[\lambda^a, P_0^{sym}\right] = \left[\lambda^a, P_1^{sym}\right] = 0\right\}. \tag{2.18}$$

2.3. *Classification of equivalence classes*

In this subsection, we will classify the equivalence classes for boundary conditions in $SU(N)$ gauge theory by using $SU(2)$ subgroup gauge transformations [9]. The matrices P_0, P_1 may not be diagonal in general. We can always diagonalize one of them, say P_0, through a global gauge transformation, but P_1 might not be diagonal. However, in Ref. [9], we know each equivalence class has (P_0, P_1) that are both diagonal representations. So, let us consider diagonal P_0, P_1, which are specified by three non-negative integers (p, q, r) such that

$$\begin{aligned} \mathtt{diag}\, P_0 &= \overbrace{(+1, \ldots, +1, +1, \ldots, +1, -1, \ldots, -1, -1, \ldots, -1)}^{N} \\ \mathtt{diag}\, P_1 &= (\underbrace{+1, \ldots, +1}_{p}, \underbrace{-1, \ldots, -1}_{q}, \underbrace{+1, \ldots, +1}_{r}, \underbrace{-1, \ldots, -1}_{s=N-p-q-r}), \end{aligned} \tag{2.19}$$

where $N \geqslant p, q, r, s \geqslant 0$. We denote the boundary conditions indicated (p, q, r) as $[p; q, r; s]$. The matrix P_0 is interchanged with P_1 by the interchange of q and r. To illustrate the boundary-changing local gauge transformations, we consider an $SU(2)$ gauge theory with $(P_0, P_1, U) = (\tau_3, \tau_3, I)$. After gauge transformation $\Omega = \exp\{i(\frac{\alpha y}{2\pi R})\tau_2\}$, we obtain the equivalence relation

$$(\tau_3, \tau_3, I) \sim \left(\tau_3, e^{i\alpha\tau_2}\tau_3, e^{i\alpha\tau_2}\right). \tag{2.20}$$

In particular, for $\alpha = \pi$ we have

$$(\tau_3, \tau_3, I) \sim (\tau_3, -\tau_3, -I). \tag{2.21}$$

Using this equivalence relation, we can have the following equivalence relations in $SU(N)$ gauge theory:

$$\begin{aligned} [p, q, r, s] &\sim [p-1; q+1, r+1; s-1] \quad \mathtt{for}\ p, s \geqslant 1 \\ &\sim [p+1; q-1, r-1; s+1] \quad \mathtt{for}\ q, r \geqslant 1 \end{aligned} \tag{2.22}$$

The sets of boundary conditions connected by this equivalence relations lead to the same physical content. We can completely classify the equivalence classes in $SU(N)$ gauge theory on orbifold, by using (2.19), (2.22). It has been showed that the number of equivalence classes in $SU(N)$ gauge theory on orbifold equals to $(N+1)^2$ [9].

3. Reformulation of gauge-Higgs unification with dynamical boundary conditions

In this section, we will give a formulation for GHU including the dynamics of boundary conditions, and show only restricted sets of boundary conditions practically contribute to the partition function.

3.1. Definition of model

The partition function for $SU(N)$ GHU on orbifold is given by:

$$Z = \int_C dP_0 \int_C dP_1 \int \mathcal{D}A_M \mathcal{D}\bar{\psi}\mathcal{D}\psi\Big|_{P_0, P_1} e^{iS(A_M, \psi, P_0, P_1)}, \tag{3.1}$$

where,

$$C = \left\{P_i \in U(N),\ P_i^2 = 1\right\} \quad i = 1, 2 \tag{3.2}$$

and $S(A_M, \psi, P_0, P_1)$ is the action depending on gauge fields, fermion fields and boundary conditions. We suppose that the action $S(A_M, \psi, P_0, P_1)$ is invariant under gauge transformation on fields A_M, ψ, but the boundary conditions may not be so. The symbol $|_{P_0, P_1}$ means we restrict functional integral regions for fields A_M, ψ to preserve the boundary conditions. dP_0, dP_1 are defined as $U(N)$ invariant measures.

3.2. Natures of integral with dP_0, dP_1

We will discuss general natures of integration over the boundary conditions $\int_C dP_0 \int_C dP_1$. First, we consider the following transformation for integral variable P_0

$$P_0 = U^\dagger P_0' U, \tag{3.3}$$

where $U \in U(N)$. Under this transformation, the integration over P_0 converts into

$$\begin{aligned}
&\int\limits_C dP_0 = \int\limits_{C'} d\big[U^\dagger P_0' U\big] \\
&C \equiv \big\{P_0 \in U(N),\ P_0^2 = 1\big\} \\
&C' = UCU^\dagger.
\end{aligned} \tag{3.4}$$

Note $d[U^\dagger P_0' U] = dP_0'$ from the property of invariant measure. Since $(P_0')^2 = 1,\ P_0' \in C'$, we can see $C = C'$. So, we find

$$\int\limits_C dP_0 = \int\limits_C dP_0'. \tag{3.5}$$

The same discussion can apply to P_1.

Next, we will give the method which splits integration of a function depending on P_0, P_1 between diagonal variables and off-diagonal variables [10,11]. We start with

$$\mathrm{F} = \int\limits_C dP_0 \int\limits_C dP_1 f(P_0, P_1), \tag{3.6}$$

where $f(P_0, P_1)$ is a function depending on P_0, P_1, and we assume $f(P_0, P_1)$ is invariant under transformation $P_0 \to UP_0U^\dagger$, $P_1 \to UP_1U^\dagger$ $U \in U(N)$. That is,

$$f\big(UP_0U^\dagger, UP_1U^\dagger\big) = f(P_0, P_1). \tag{3.7}$$

Then, we define the following function

$$\begin{aligned}
&\Delta^{-1}(P_0) \equiv \int dU \prod_{1\leqslant i<j\leqslant N} \delta^{(2)}\big[\big(UP_0U^\dagger\big)_{ij}\big] \\
&\delta^{(2)}\big[\big(UP_0U^\dagger\big)_{ij}\big] \equiv \delta\big[\Re\big(UP_0U^\dagger\big)_{ij}\big]\delta\big[\Im\big(UP_0U^\dagger\big)_{ij}\big].
\end{aligned} \tag{3.8}$$

Here, dU is the invariant measure of $U(N)$. Substituting the function defined by (3.8) to (3.6), we find

$$\mathrm{F} = \int\limits_C dP_0 \int\limits_C dP_1\, \Delta(P_0) \int dU \prod_{1\leqslant i<j\leqslant N} \delta^{(2)}\big[\big(UP_0U^\dagger\big)_{ij}\big] f(P_0, P_1). \tag{3.9}$$

Change the variable as $P_0 = U^\dagger P_0' U$. Since the function (3.8) is invariant under this transformation, and by using (3.5), we find

$$\mathrm{F} = \int dU \int\limits_C dP_0' \int\limits_C dP_1\, \Delta\big(P_0'\big) \prod_{1\leqslant i<j\leqslant N} \delta^{(2)}\big[\big(P_0'\big)_{ij}\big] f\big(U^\dagger P_0' U, P_1\big). \tag{3.10}$$

Change the variable as $P_1 = U^\dagger P_1' U$, and using (3.5) where P_0 is replaced with P_1 and (3.7), (3.10) equals to

$$\mathrm{F} = \int dU \int_C dP_0' \int_C dP_1' \Delta(P_0') \prod_{1 \leqslant i < j \leqslant N} \delta^{(2)}\big[(P_0')_{ij}\big] f(P_0', P_1'). \tag{3.11}$$

We normalize $\int dU = 1$. Next, we change the integral region by regularization parameter μ to regularize (3.11).

$$C \to \hat{C} \equiv \{P_0 \in U(N),\ \rho_i = \pm e^{i\mu_i},\ 0 \leqslant \mu_i \leqslant \mu \ll 1\} \quad \mu\text{: real}. \tag{3.12}$$

ρ_i $(1 \leqslant i \leqslant N)$ denote eigenvalues of P_0. This change preserves the relation (3.5), and in the limit $\mu \to 0$ we can restore it to the original definition. At the end of our calculation, we must take the limit $\mu \to 0$. Carry out integration of P_0' with δ function in (3.10), it becomes

$$\mathrm{F} = \int d\Lambda_0 \int_C dP_1' \, \Delta(\Lambda_0) f(\Lambda_0, P_1'), \tag{3.13}$$

where

$$\Delta^{-1}(\Lambda_0) = \frac{(2\pi)^N}{\prod_{1 \leqslant i < j \leqslant N} |\epsilon_i - \epsilon_j e^{i\mu_{ij}}|^2}, \quad \mu_{ij} = \mu_j - \mu_i. \tag{3.14}$$

ϵ_i, ϵ_j are $+1$ or -1.

The symbol $\int d\Lambda_0$ denotes integration over only diagonal matrices in the integral region $\hat{C}$. In the regularization (3.12), it is represented by

$$\int d\Lambda_0 = \sum_{\pm 1} \int_0^\mu \prod_{1 \leqslant n \leqslant N} d\mu_n. \tag{3.15}$$

$\sum_{\pm 1}$ means the summation over all combinations we assign $+1$ or -1 to ϵ_i $(1 \leqslant i \leqslant N)$ in (3.13), (3.14).

We can apply the same calculation and regularization from (3.8) to (3.14) for P_1, and we have

$$\mathrm{F} = \int d\Lambda_0 \int d\Lambda_1 \, \Delta(\Lambda_0)\Delta(\Lambda_1) \int dU \, f(\Lambda_0, U^\dagger \Lambda_1 U), \tag{3.16}$$

where,

$$\Delta^{-1}(\Lambda_1) = \frac{(2\pi)^N}{\prod_{1 \leqslant p < q \leqslant N} |\epsilon_p' - \epsilon_q' e^{i\mu_{pq}'}|^2}, \quad \mu_{pq}' = \mu_q' - \mu_p' \tag{3.17}$$

$$\int d\Lambda_1 = \sum_{\pm 1} \int_0^{\mu'} \prod_{1 \leqslant m \leqslant N} d\mu_m' \tag{3.18}$$

$\mu' \ll 1$ is the regularization parameter, and ϵ_p', ϵ_q' are $+1$ or -1.

For boundary conditions $P_0, P_1 \in U(N)$, $N \geqslant 3$, taking the limit $\mu, \mu' \to 0$ in (3.16) lead to $\mathrm{F} \to 0$. It means the integral regions for the boundary conditions correspond to the regions of measure zero in the $U(N)$ invariant measure in $U(N)$ group manifold. We must renormalize the partition function (3.1) to make it well-defined.

3.3. Integration of partition function for boundary conditions

In this subsection, we will apply the method discussed in Subsection 3.2 to the model defined in Subsection 3.1. We will find that only some of sets of boundary conditions practically contribute to the partition function. First, as noted in the end of Subsection 3.2 we need to divide the partition function (3.1) by the volume $\int_C dP_0 \int_C dP_1$. We regularize the integral $\int_C dP_0 \int_C dP_1$ in the denominator and numerator with parameters μ, μ', and adopt the following normalization when we take the limit,

$$\frac{\int_{\hat{C}} dP_0 \int_{\hat{C}} dP_1}{\int_{\hat{C}} dP_0 \int_{\hat{C}} dP_1} \to 1, \qquad \mu, \mu' \to 0. \tag{3.19}$$

According to the discussion in Subsection 3.2, this volume can be written as

$$V \equiv \int\limits_{\hat{C}} dP_0 \int\limits_{\hat{C}} dP_1 = \int d\Lambda_0 \int d\Lambda_1 \, \Delta(\Lambda_0)\Delta(\Lambda_1). \tag{3.20}$$

The notations follow the definitions in Subsection 3.2. The normalized partition function, Z, is defined by

$$\mathrm{Z} = V^{-1} \int\limits_{\hat{C}} dP_0 \int\limits_{\hat{C}} dP_1 \int \mathcal{D}A_M \mathcal{D}\bar{\psi}\mathcal{D}\psi \bigg|_{P_0, P_1} e^{iS(A_M, \psi, P_0, P_1)}. \tag{3.21}$$

The field values are not defined in this regularization since $P_0^2, P_1^2 \neq 1$. We redefine parity transformation matrices $\hat{P}_0, \hat{P}_1$ as

$$\hat{P}_0 \equiv \left(P_0^{-2}\right)^{\frac{1}{2}} P_0, \qquad \hat{P}_1 \equiv \left(P_1^{-2}\right)^{\frac{1}{2}} P_1, \tag{3.22}$$

where

$$A^{\frac{1}{2}} = U\Lambda^{\frac{1}{2}}U^\dagger, \qquad \Lambda^{\frac{1}{2}} = \begin{pmatrix} \sqrt{a_1} & & \\ & \sqrt{a_2} & \\ & & \ddots \end{pmatrix} \quad A \in U(N). \tag{3.23}$$

a_i $(i = 1, 2, \ldots)$ are the eigenvalues of A, and we choose the positive square root of the eigenvalues as the convention. In this prescription, we find the eigenvalues of $\hat{P}_0, \hat{P}_1$ are $+1$ or -1, and $\hat{P}_0{}^2 = \hat{P}_1{}^2 = 1$. We can restore those to the original definitions in the limit $\mu, \mu' \to 0$. The integrand of the boundary conditions is well-defined function. From now on, the symbol $|_{P_0, P_1}$ means we restrict the functional integral regions for fields A_M, ψ to preserve the boundary conditions $\hat{P}_0, \hat{P}_1$.

The next step is to divide the integration in the partition function into diagonal components and off-diagonal components of boundary condition matrices P_0, P_1, according to Subsection 3.2.

$$\mathrm{Z} = V^{-1} \int\limits_{\hat{C}} dP_0 \int\limits_{\hat{C}} dP_1 \int \mathcal{D}A_M \mathcal{D}\bar{\psi}\mathcal{D}\psi \bigg|_{P_0, P_1} \Delta(P_0) \int dU \, \delta^{(2)}\left(UP_0U^\dagger\right) e^{iS(A_M, \psi, P_0, P_1)}, \tag{3.24}$$

where,

$$\delta^{(2)}\left(UP_0U^\dagger\right) \equiv \prod_{1\leqslant i<j\leqslant N} \delta^{(2)}\left[\left(UP_0U^\dagger\right)_{ij}\right]. \tag{3.25}$$

In (3.24), we change the integration variable from P_0 to $P_0' = UP_0U^\dagger$, and using (3.5), we find

$$Z = V^{-1}\int dU \int_{\hat{C}} dP_0' \int_{\hat{C}} dP_1 \int \mathcal{D}A_M\mathcal{D}\bar{\psi}\mathcal{D}\psi\Big|_{U^\dagger P_0'U,P_1} \Delta\left(P_0'\right)\delta^{(2)}\left(P_0'\right) \\ \times e^{iS(A_M,\psi,U^\dagger P_0'U,P_1)}. \tag{3.26}$$

Following the discussion in Subsection 3.2, we integrate out $\int dP_0'$ in (3.26) with $\delta^{(2)}(P_0')$. Eq. (3.26) equals to

$$Z = V^{-1}\int dU \int d\Lambda_0 \int_{\hat{C}} dP_1 \Delta(\Lambda_0) \int \mathcal{D}A_M\mathcal{D}\bar{\psi}\mathcal{D}\psi\Big|_{U^\dagger\Lambda_0U,P_1} e^{iS(A_M,\psi,U^\dagger\Lambda_0U,P_1)}. \tag{3.27}$$

$U^\dagger\Lambda_0U$ is the unitary transformation for $U \in U(N)$. But we can regard this transformation as the unitary transformation for $U' \in SU(N)$. One can multiply this transformation by diagonal $U(N)$ element Λ, as $U^\dagger\Lambda_0U$ preserves its value. That is, $U'^\dagger\Lambda_0U' = U^\dagger\Lambda_0U$ for $U' = \Lambda U$. So, by multiplying U by suitable Λ, we can find $U' = \Lambda U$ $U' \in SU(N)$ for arbitrary $U \in U(N)$. Therefore, we can rewrite (3.27) as

$$Z = V^{-1}\int dU \int d\Lambda_0 \int_{\hat{C}} dP_1 \Delta(\Lambda_0) \int \mathcal{D}A_M\mathcal{D}\bar{\psi}\mathcal{D}\psi\Big|_{U'^\dagger\Lambda_0U',P_1} e^{iS(A_M,\psi,U'^\dagger\Lambda_0U',P_1)} \\ U' \in SU(N). \tag{3.28}$$

Change the integration variable from P_1 to $P_1' = U'P_1U'^\dagger$ and use (3.5) where P_0 is replaced with P_1, We have

$$Z = V^{-1}\int dU \int d\Lambda_0 \int_{\hat{C}} dP_1' \Delta(\Lambda_0) \int \mathcal{D}A_M\mathcal{D}\bar{\psi}\mathcal{D}\psi\Big|_{U'^\dagger\Lambda_0U',U'^\dagger P_1'U'} \\ \times e^{iS(A_M,\psi,U'^\dagger\Lambda_0U',U'^\dagger P_1'U')}. \tag{3.29}$$

Eq. (3.29) equals to the original system that has the boundary conditions $(\hat{\Lambda}_0, \hat{P}_1')$ up to the global gauge transformation U'. The system should be independent on global gauge. So, the relation (3.29) becomes

$$Z = V^{-1}\int dU \int d\Lambda_0 \int_{\hat{C}} dP_1' \Delta(\Lambda_0) \int \mathcal{D}A_M\mathcal{D}\bar{\psi}\mathcal{D}\psi\Big|_{\Lambda_0,P_1'} e^{iS(A_M,\psi,\Lambda_0,P_1')}. \tag{3.30}$$

Normalize $\int dU = 1$, and apply in the same procedure to P_1'. Then, Eq. (3.30) becomes

$$Z = \frac{\sum_{\pm 1}\int_0^{\mu}\prod_{1\leqslant n\leqslant N} d\mu_n \int_0^{\mu'}\prod_{1\leqslant m\leqslant N} d\mu_m' \, \Delta(\Lambda_0)\Delta(\Lambda_1)I(A_M,\psi,\Lambda_0,\Lambda_1)}{\sum_{\pm 1}\int_0^{\mu}\prod_{1\leqslant n'\leqslant N} d\mu_{n'} \int_0^{\mu'}\prod_{1\leqslant m'\leqslant N} d\mu_{m'}' \, \Delta(\Lambda_0)\Delta(\Lambda_1)} \tag{3.31}$$

where

$$I(A_M,\psi,\Lambda_0,\Lambda_1)\equiv\int dU\int \mathcal{D}A_M\mathcal{D}\bar{\psi}\mathcal{D}\psi\bigg|_{\Lambda_0,U^\dagger\Lambda_1 U} e^{iS(A_M,\psi,\Lambda_0,U^\dagger\Lambda_1 U)}. \tag{3.32}$$

We suppose $I(A_M,\psi,\Lambda_0,\Lambda_1)$ is almost constant function on the integral variables μ_n and μ'_m, compared with $\Delta(\Lambda_0),\Delta(\Lambda_1)$. Then, we can replace the function $\int_0^{\mu}\prod_{1\leqslant n\leqslant N}d\mu_n\times\int_0^{\mu'}\prod_{1\leqslant m\leqslant N}d\mu'_m\,\Delta(\Lambda_0)\Delta(\Lambda_1)$ with the integrand on particular values μ_n,μ'_m ($0<\mu_n$, $\mu'_m<\mu$) times the integral regions by mean-value theorem. We can put the conditions $\mu_{ij}\neq 0$, $\mu'_{pq}\neq 0$, $(1\leqslant i,p<j,q\leqslant N)$ if ϵ_i and ϵ_j or ϵ'_p and ϵ'_q have the same sign, since these values correspond to the maximum or minimum of the integrand. After this replacement, the integral regions of $d\mu_n, d\mu'_m$ between the denominator and numerator in (3.31) cancel out. As a result, we have

$$Z=\frac{\sum_{\pm 1}\prod_{1\leqslant i,p<j,q\leqslant N}|\epsilon_i-\epsilon_j e^{i\mu_{ij}}|^2|\epsilon'_p-\epsilon'_q e^{i\mu'_{pq}}|^2 I(A_M,\psi,\Lambda_0,\Lambda_1)}{\sum_{\pm 1}\prod_{1\leqslant k,v<l,w\leqslant N}|\epsilon_k-\epsilon_l e^{i\mu_{kl}}|^2|\epsilon'_v-\epsilon'_w e^{i\mu'_{vw}}|^2}. \tag{3.33}$$

In the summation $\sum_{\pm 1}$, the factors $|1-e^{i\mu_{ij}}|^2$ and $|1-e^{i\mu'_{pq}}|$ give 0 to each term in (3.32) when we take the limit $\mu,\mu'\to 0$. We suppose "a" is the lowest number of the factors, such as $|1-e^{i\mu_{ij}}|$, each term has in (3.33). We can find the lowest number of the factors such as $|1-e^{i\mu'_{pq}}|$ is also a. Then, we multiply the denominator and numerator in (3.33) by $|1-e^{i\mu}|^{-2a}$. As taking the limit $\mu\to 0$, we can see

$$\frac{|1-e^{i\mu_{ij}}|}{|1-e^{i\mu}|}=\left|\frac{\mu_{ij}}{\mu}\right|\to C_{ij}>0. \tag{3.34}$$

C_{ij} must be finite value in order to be consistent with mean-value theorem. There is at least one term which has finite value in denominator and numerator of (3.33) in this limit. Such finite terms correspond to the terms which have the highest number of pairs of different signs substituted for ϵ_i $(1\leqslant i\leqslant N)$. The other terms go to 0 when we take the limit $\mu\to 0$. Since the ϵ_i $(1\leqslant i\leqslant N)$ denote the eigenvalues of P_0, we can find only sets of the eigenvalues of P_0 which have the highest number of pairs of different signs contribute to the partition function in (3.33). We will have the same conclusion if previous discussion is applied to the integral of P_1.

Relating these results to the discussion about the dimensions of unitary conjugate class for a particular set of eigenvalues gives us more observations. In $U(N)$ group if a set of eigenvalues has no the identical eigenvalue, the submanifold which consists of the elements of unitary conjugate class for its set of eigenvalues has the highest dimensions among the submanifolds of the unitary conjugate classes. And the more identical eigenvalues a set of eigenvalues includes, the less dimensions the submanifold of its unitary conjugate class has [12]. Therefore, in our case, unitary conjugate classes which includes the highest number of the pairs $+1$, -1 as the eigenvalues of P_0, P_1 have the highest dimensions among the sets of eigenvalues of P_0, P_1, and only these boundary conditions contribute to the partition function in the integral process.

Next, let us consider the case that boundary conditions (Λ'_0,Λ'_1) are related to the diagonal boundary conditions (Λ_0,Λ_1) by permutation of the eigenvalues sets. We will show $I(A_M,\psi,\Lambda'_0,\Lambda'_1)$ gives an identical contribution to the partition function as $I(A_M,\psi,\Lambda_0,\Lambda_1)$. Since (Λ'_0,Λ'_1) is the permutation of eigenvalues sets in (Λ_0,Λ_1), it satisfies the relations

$$\Lambda'_0=V_0^\dagger\Lambda_0 V_0,\qquad \Lambda'_1=V_1^\dagger\Lambda_1 V_1,\qquad V_0,V_1\in SU(N), \tag{3.35}$$

and the factors $\prod_{1\leqslant i<j\leqslant N}|\epsilon_i-\epsilon_j e^{i\mu_{ij}}|^2$, $\prod_{1\leqslant p<q\leqslant N}|\epsilon'_p-\epsilon'_q e^{i\mu'_{pq}}|^2$ give identical contribution to $I(A_M,\psi,\Lambda'_0,\Lambda'_1)$ and $I(A_M,\psi,\Lambda_0,\Lambda_1)$. One find for the boundary conditions (Λ'_0,Λ'_1) in (3.32)

$$
\begin{aligned}
&I\left(A_M, \psi, \Lambda_0', \Lambda_1'\right) \\
&\quad = \int dU \int \mathcal{D}A_M \mathcal{D}\bar{\psi}\mathcal{D}\psi \bigg|_{\Lambda_0', U^\dagger \Lambda_1' U} e^{iS(A_M, \psi, \Lambda_0', U^\dagger \Lambda_1' U)} \\
&\quad = \int dU \int \mathcal{D}A_M \mathcal{D}\bar{\psi}\mathcal{D}\psi \bigg|_{V_0^\dagger \Lambda_0 V_0, U^\dagger V_1^\dagger \Lambda_1 V_1 U} e^{iS(A_M, \psi, V_0^\dagger \Lambda_0 V_0, U^\dagger V_1^\dagger \Lambda_1 V_1 U)}.
\end{aligned}
\tag{3.36}
$$

Under global gauge transformation $\Lambda_0' \to V_0 \Lambda_0' V_0^\dagger$, $U^\dagger \Lambda_1' U \to V_0 U^\dagger \Lambda_1' U V_0^\dagger$, we find

$$
\begin{aligned}
&I(A_M, \psi, \Lambda_0', \Lambda_1') \\
&\quad = \int dU \int \mathcal{D}A_M \mathcal{D}\bar{\psi}\mathcal{D}\psi \bigg|_{\Lambda_0, V_0 U^\dagger V_1^\dagger \Lambda_1 V_1 U V_0^\dagger} e^{iS(A_M, \psi, \Lambda_0, V_0 U^\dagger V_1^\dagger \Lambda_1 V_1 U V_0^\dagger)}.
\end{aligned}
\tag{3.37}
$$

Using the property of $\int dU$ invariant measure, we have

$$
I\left(A_M, \psi, \Lambda_0', \Lambda_1'\right) = I(A_M, \psi, \Lambda_0, \Lambda_1). \tag{3.38}
$$

Then, we can see in (3.38) $I(A_M, \psi, \Lambda_0, \Lambda_1)$ and $I(A_M, \psi, \Lambda_0', \Lambda_1')$ give the identical contributions to (3.31). According to the discussion in Subsection 2.3, there is the gauge transformation which relates the boundary conditions (Λ_0, Λ_1) to (Λ_0', Λ_1'). Then, it is worthwhile to state the boundary conditions (Λ_0, Λ_1) and (Λ_0', Λ_1') are in the same equivalence class. According to the discussion of Appendix A in Ref. [9], we can see there is at least one both diagonal boundary conditions in each equivalence class. Then, on the process that arbitrary boundary conditions change to both diagonal representations by global and local gauge transformations, there is no transformation which changes the eigenvalues set of the boundary conditions. So, arbitrary boundary conditions (P_0, P_1) and its eigenvalue set (Λ_0, Λ_1) belong to the same equivalence class. Since a permutation (Λ_0', Λ_1') of diagonal representations (Λ_0, Λ_1) belong to the equivalence class with (Λ_0, Λ_1), we conclude equivalence classes for GHU on S^1/Z_2 in $SU(N)$ gauge theory are completely classified by eigenvalues sets for boundary conditions. Therefore, on the process that we compute some physical observables, the integrand on $\int dU$ in (3.31) is independent of the variable U, so it is sufficient to compute only about the both diagonal representations $(P_0, P_1) = (\Lambda_0, \Lambda_1)$ if we want to know some physical observables.

4. Application to several examples

In this section, we apply the formulation we had in Section 3 to $SU(2)$, $SU(3)$, $SU(5)$ gauge theory. In particular, we are interested in $SU(5)$ gauge theory as the candidate for GUT. As a consequence of the boundary conditions dynamics presented here, sets of the boundary conditions will be highly restricted. In $SU(5)$ case, we will show these restricted sets include the equivalence classes which have the standard model symmetry $SU(3) \times SU(2) \times U(1)$ as the symmetry of boundary conditions.

First, we consider $SU(2)$ gauge theory on $M^4 \times S^1/Z_2$ as the simplest example. In the case, there is only one equivalence class of boundary conditions that gives a non-vanishing contribution to the partition function. It is characterized by the eigenvalue set

$$
\left\{ \begin{array}{l} P_0 = \{+1, -1\} \\ P_1 = \{+1, -1\} \end{array} \right\} \tag{4.1}
$$

This boundary conditions lead to the symmetry breaking $SU(2) \to U(1)$ as symmetry of boundary conditions.

The next example is $SU(3)$ gauge theory. In this case, the four sets of boundary conditions and their equivalence classes contribute to the partition function. These equivalence classes are characterized by following eigenvalue sets

$$\begin{aligned} &(1) && (2) \\ &\begin{Bmatrix} P_0 = \{+1, +1, -1\} \\ P_1 = \{+1, +1, -1\} \end{Bmatrix} && \begin{Bmatrix} P_0 = \{+1, +1, -1\} \\ P_1 = \{+1, -1, -1\} \end{Bmatrix} \\ &(3) && (4) \\ &\begin{Bmatrix} P_0 = \{+1, -1, -1\} \\ P_1 = \{+1, +1, -1\} \end{Bmatrix} && \begin{Bmatrix} P_0 = \{+1, -1, -1\} \\ P_1 = \{+1, -1, -1\} \end{Bmatrix}. \end{aligned} \tag{4.2}$$

The boundary conditions (1) and (4) lead to the symmetry breaking $SU(3) \to SU(2) \times U(1)$. On the other hand, the boundary conditions (2) and (3) lead to the symmetry breaking $SU(3) \to U(1) \times U(1)$. The partition function in (3.33) is written as

$$\mathrm{Z} = C_1 I_{(1)} + C_2 I_{(2)} + C_3 I_{(3)} + C_4 I_{(4)}. \tag{4.3}$$

Here, $I_{(i)}$ $i = 1 \sim 4$ indicate the $I(A_M, \psi, \Lambda_0, \Lambda_1)$ in (3.32). The subscript indices mean we substitute corresponding boundary conditions (i) $i = 1 \sim 4$ in (4.2) into $I_{(i)}$ respectively.

Since the factors $\prod_{1 \leqslant i < j \leqslant N} |\epsilon_i - \epsilon'_j e^{i\mu_{ij}}|^2$, $\prod_{1 \leqslant p < q \leqslant N} |\epsilon'_p - \epsilon'_q e^{i\mu'_{pq}}|^2$ give the overall constant in (4.3), we dropped this constant. C_i denote the coefficients corresponding to all permutation in the boundary conditions (i) $i = 1 \sim 4$. In $SU(3)$ case, these constants are

$$C_i = \left({}^3C_1\right)^2 \quad i = 1 \sim 4. \tag{4.4}$$

So, we can see all coefficients are the same, and drop this coefficients as overall constants.

Finally, we investigate $SU(5)$ gauge theory example. Just as in the $SU(3)$ example, four boundary conditions sets and their equivalence classes contribute to the partition function. These equivalence classes are characterized by

$$\begin{aligned} &(1) && (2) \\ &\begin{Bmatrix} P_0 = \{+1, +1, +1, -1, -1\} \\ P_1 = \{+1, +1, +1, -1, -1\} \end{Bmatrix} && \begin{Bmatrix} P_0 = \{+1, +1, +1, -1, -1\} \\ P_1 = \{+1, +1, -1, -1, -1\} \end{Bmatrix} \\ &(3) && (4) \\ &\begin{Bmatrix} P_0 = \{+1, +1, -1, -1, -1\} \\ P_1 = \{+1, +1, +1, -1, -1\} \end{Bmatrix} && \begin{Bmatrix} P_0 = \{+1, +1, -1, -1, -1\} \\ P_1 = \{+1, +1, -1, -1, -1\} \end{Bmatrix}. \end{aligned} \tag{4.5}$$

Boundary conditions (2) and (3) lead to the symmetry breaking $SU(5) \to SU(2) \times SU(2) \times U(1) \times U(1)$. We should mention the boundary conditions (1) and (4) have $SU(3) \times SU(2) \times U(1)$ standard model symmetry as the symmetry of boundary conditions. The partition function consists of the four part that correspond to the boundary conditions (1) $\sim$ (4) respectively. We note that physical symmetry depends on the matter content.

5. Conclusion

In this paper, we have supposed that fundamental theory can describe the dynamics of the boundary conditions in GHU, and have discussed the natures of the measures dP_0, dP_1. In the present scenario of GHU, the orbifold boundary conditions are imposed in an ad hoc manner among many possible choices. The boundary conditions can be classified in equivalence classes by using the Hosotani mechanism. Two theories in the same equivalence class lead to the identical physical content. In particular, the number of equivalence classes of $SU(N)$ gauge theory on $M^4 \times S^1/Z_2$ is $(N+1)^2$. In other words, $SU(N)$ gauge theory on $M^4 \times S^1/Z_2$ has $(N+1)^2$ different theories.

We have showed only the boundary conditions which have the highest number of the pair $+1$, -1 in eigenvalues of P_0, P_1 eventually contribute to the partition function in our formulation. The submanifolds which consist of these boundary conditions as the elements have the highest dimensions among submanifolds of the equivalence classes for boundary conditions. In $SU(N)$ gauge theory where N is odd, the four equivalence classes practically contribute to partition function. These equivalence classes lead to nontrivial breakdown of the symmetries imposed on Lagrangian density. To determine which set of boundary conditions is selected as physical state in these four sets of boundary conditions, we need to evaluate the effective potentials for each set of boundary conditions. But the difference between two equivalence classes may appear to be infinite. It is known that the energy difference become finite in supersymmetric GHU.

To consider the arbitrariness problem completely, we should regard GHU as an effective theory given by the more fundamental theory. The fundamental theory may select the lowest energy state as physical state by giving the dynamics of the boundary conditions.

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